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DYNAMICS OF A SIMPLEST MODEL OF A WHEELED VEHICLE UNDER RANDOM PERTURBATIONS

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The dynamics of a simplest model of a wheeled vehicle in its region of stability under a constant action of small perturbations of determinate as well as of random character is considered. Investigation of the track stability of the wheeled vehicles is usually reduced to determining the ranges of values of the parameters over which the motion of the system is stable, and to studying the character of the transition process in the region of stability. Of practical interest is the study of the dynamics of the system within the region of stability under the influence of small constantly acting perturbations of determinate as well as of random character (unevenness in the microprofile of a road and wind loads). Such an investigation enables us to estimate the spread of the coordinates relative to the values of the parameters lying within the region of stability and to solve the problems of opimizing these parameters.

1. Statement of the problem. To simplify the operations, we shall consider the case of an uncontrolled motion of a simple model of a wheeled vehicle in the horizontal plane. The equations of motion constructed with help of the Rocard drift hypothesis [1], have the form

$$\begin{aligned} x_1^{\cdot} &= -Ax_1 - Ax_2 - Bx_3 + \xi_1 + \xi_2 \\ x_2^{\cdot} &= x_3 \\ x_3^{\cdot} &= -Bx_1 - Bx_2 - Cx_3 - d_1\xi_1 - d_2\xi_2 \end{aligned} \tag{1.1}$$

Here we use the following dimensionless variables and notation:

$$\tau = \frac{v}{\rho}t, \quad x_1 = \frac{1}{\rho}\frac{dx}{d\tau} \equiv \frac{x}{\rho}, \quad x_2 = \theta, \quad x_3 = \frac{d\theta}{d\tau} \equiv \theta^{\cdot}$$

$$A = \frac{(a_1 + a_2)\rho}{mv^2}, \quad B = \frac{a_2l_2 - a_1l_1}{mv^2}, \quad C = \frac{a_1l_1^2 + a_2l_2^2}{\rho mv^2}$$

$$d_1 = \frac{l_1}{\rho}, \quad d_2 = -\frac{l_2}{\rho}, \quad \xi_1 = \frac{\rho F_1}{mv^2}, \quad \xi_2 = \frac{\rho F_2}{mv^2}$$

We assume that the unperturbed motion of the vehicle takes place on the xy-plane along the y-axis, θ is the angle between the longitudinal axis of the vehicle and the y-axis, t is time, v is the velocity of motion along the y-axis, m is the mass of the vehicle, ρ is the radius of inertia of the vehicle relative to the vertical axis passing through the center of mass, a_1 and a are the drift coefficients of the front and rear wheels, l_1 and l_2 are the distances between the center of mass and the front and rear axle, $L = l_1 + l_2$ is the wheelbase of the vehicle, and F_1 and F_2 are given perturbations acting of the front and rear wheels.

In the dimensionless variables the set of steady motions becomes a set of equilibrium states, at each point of which the relation $x_1 + x_2 = 0$ holds.

The characteristic polynomial has the form

$$p\,[p^2 + p\,(A + C) + AC + B - B^2] = p\Delta\,(p)$$

where p denotes a dimensionless differentiation operator. Neglecting the null root arising as the result of the one-dimensional character of the set [1], we find the conditions of stability of this set

$$A + C > 0, \quad AC + B - B^2 > 0$$

In accordance with the physical sense of the parameters of the system, the first condition always holds, and the second condition can be reduced to the following inequality:

$$v^2 < \frac{a_1 a_2 L^2}{m\,(a_1 l_1 - a_2 l_2)} \tag{1.2}$$

which in fact defines the range of the values of the parameters of the stable system's motion.

Let us now inspect the behavior of the system within the region of stability. Applying Laplace transformation to Eqs. (1.1), we obtain

$$X_k(p) = \lambda_{k1}\xi_1(p) + \lambda_{k2}\xi_2(p) \quad (k = 1, 2), \quad X_3(p) = pX_2(p)$$

$$\lambda_{1i} = \frac{p^2 + p\,(C + Bd_i) + B + Ad_i}{p\Delta\,(p)}, \quad \lambda_{2i} = -\frac{B + d_i\,(p + A)}{p\Delta\,(p)} \quad (i = 1, 2)$$

The transfer functions obtained for X_1 and X_2 contain a zero root in the denominator. From this it follows that when a unit step impulse $\xi_1 = 1\,(\tau)$ acts on the system under the condition that $\xi_2 = k\xi_1$, the solution has the following structure:

$$X_{1,2}(\tau) = \pm\frac{B + Ad_1 + k\,(B + Ad_2)}{AC + B - B^2} + \sum_{i=1}^{2} A_i^{(1,2)} \exp(p_i\,\tau)$$

From this we see that in the region of stability of the system, the representative point approaches asymptotically a position different from the initial position when $\tau \to \infty$ The expressions obtained imply also that when the relation $a_2 = ka_1$ connecting the drift coefficients of the front and rear wheels holds, then stabilization of the vehicle moving on course under the action of sidewise perturbations becomes feasible. In the case when the sidewise perturbations are due to wind loads, the coefficient k becomes a constant determined by a vehicle profile.

Let us now investigate the solution of the system under random perturbations, and let the probabilistic characteristics of the perturbations $\xi_1(\tau)$ and $\xi_2(\tau)$ be given. We determine the dispersions of the coordinates x_i, $i = 1, 2, 3$. We know from the theory of random processes [2] that the dispersion D_x of the coordinate x of the system can be expressed, in the presence of two random influences, by an expression of the form

$$D_x = \sum_{i=1}^{2}\sum_{r=1}^{2}\int_{-\infty}^{\infty} W_i(j\omega)\, W_r(-j\omega)\, S_{\xi_i\xi_r}(\omega)\, d\omega$$

where $S_{\xi_i\xi_r}(\omega)$ $(i, r = 1, 2)$ are given spectral densities and reciprocal spectral densities of the processes $\xi_1(\tau)$ and $\xi_2(\tau)$, while $W_1(j\omega)$ and $W_2(j\omega)$ are the transfer functions operating from ξ_1 and ξ_2 to the coordinate x, respectively.

The presence of zero roots in the characteristic polynomial, and consequently in the denominators of the transfer functions, makes their use impossible as the dispersions turn out to be divergent. Consequently the problem has been incorrectly formulated. It can however become correct if we pass to the coordinates expressing the deviation of the representative point from the surface of equilibrium states.

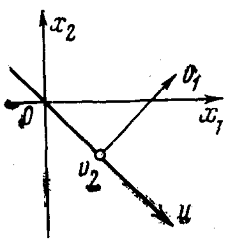

Fig. 1

Let $x_1 = v_1 - u$, $x_2 = v_1 + u$, and $x_3 = v_2$ where u is the coordinate counted along the set O_1, while v_1 and v_2 are coordinates orthogonal to u (see Fig. 1). Equations (1.1) in these variables become

$$v_1^{\cdot} = -Av_1 + \frac{1-B}{2} v_2 + \zeta_1 + \zeta_2 \quad \left(\zeta_{1,2} = \frac{1}{2}\xi_{1,2}\right) \tag{1.3}$$

$$v_2^{\cdot} = -2Bv_1 - Cv_2 - 2d_1\zeta_1 - 2d_2\zeta_2$$

$$u^{\cdot} = Av_1 + \frac{1+B}{2} v_2 - \zeta_1 - \zeta_2$$

We see that the first two equations of (1.3) form a closed system and can be considered separately. The third equation determines the motion of the representative point along the set. Applying the Laplace transformation, we obtain

$$V_i(p) = \frac{\Delta_{i1}(p)}{\Delta(p)} \zeta_1(p) + \frac{\Delta_{i2}(p)}{\Delta(p)} \zeta_2(p) \quad (i = 1, 2) \tag{1.4}$$

$$pU(p) = \frac{\Delta_{u_1}(p)}{\Delta(p)} \zeta_1(p) + \frac{\Delta_{u_2}(p)}{\Delta(p)} \zeta_2(p)$$

$$\Delta_{1j} = p + C + d_j(B-1), \quad \Delta_{2j} = -2(d_jp + d_jA + B)$$

$$\Delta_{uj} = p[A - (B+1)d_j] + AC - 2Ad_j - B(B+1) - 1 \qquad (j = 1, 2)$$

Thus the change of the coordinates (u, v_1, v_2) produced the solution (1.4) in the variables v_1 and v_2 in which the zero root no longer appears. In addition, the variable u can be determined by quadrature.

Let random forces with mathematical expectations $\langle \zeta_1 \rangle$ and $\langle \zeta_2 \rangle$ act upon the system. Then the mathematical expectations of the coordinates can be found from the equations of motion

$$\langle v_1 \rangle = \frac{C(\langle\zeta_1\rangle + \langle\zeta_2\rangle) - (1-B)(d_1\langle\zeta_1\rangle + d_2\langle\zeta_2\rangle)}{AC + B(1-B)}$$

$$\langle v_2 \rangle = -\frac{A(d_1\langle\zeta_1\rangle + d_2\langle\zeta_2\rangle) + 2B(\langle\zeta_1\rangle + \langle\zeta_2\rangle)}{AC + B(1-B)}$$

Subtracting from (1.3) the corresponding averaged equations, we obtain the equations of motion in centralized coordinates. These equations fully coincide with (1.3), provided that the coordinates in the latter equations are regarded as centralized. We shall therefore assume from now on, that v_1 and v_2 are centralized.

2. Dynamics of a wheeled vehicle under a random wind load. We shall assume for simplicity that the side force acting on the rear wheel is proportional to the force acting on the front wheel, i.e., $\xi_2 = k\xi_1$. The physical realization of this assumption can be accomplished by a side wind when the load distribution on the front and rear wheels is determined by treating the vehicle as a sail of a certain area.

Let the function $\xi_1(\tau)$ be defined by its mathematical expectation $\langle \xi_1 \rangle = \rho \langle F_1 \rangle / 2mv^2$ and the spectral density [3]

$$S_\xi(p) = \frac{R_\xi(0)\beta}{2\pi(p^2+\beta^2)}$$

The parameters $R_\xi(0)$ and β are connected with the corresponding wind parameters $R_F(0)$ and β_1 in the real time scale by the relations

$$R_\xi(0) = R_F(0)\rho^2/(4m^2v^4), \quad \beta = \rho\beta_1/v$$

In this case the dispersion of the coordinate v_1 is given by the expression

$$D_{v_1} = \int_{-\infty}^{\infty} \frac{|\Delta_{v_1}(\omega)|^2}{|\Delta(\omega)} S_\xi(\omega)\cdot d\omega$$

Substituting the expression for $S_\xi(\omega)$ in the above equation, we obtain

$$D_{v_1} = \frac{R_\xi(0)\beta}{2\pi} \int_{-\infty}^{\infty} \frac{|\Delta_{v_1}(\omega)|^2}{|\Delta(\omega)(p+\beta)|^2} (\xi)\, d\omega$$

$$\Delta_{v_1}(p) = (1+k)(p+C_1), \quad C_1 = C + \frac{(B-1)(l_1-kl_2)}{\rho(1+k)}, \quad p = j\omega$$

As we know [2], the above integral is equal to $(-1)^{n+1}M_n / 2\Delta_n$ where Δ_n is the leading Hurwitz determinant of the polynomial $\Delta(p)(p+\beta)$ appearing in the denominator of the integrand function and M_n represents the determinant Δ_n in which the first column has been replaced by the coefficients of the polynomial

$\Delta_{v_1}(p)$. To determine the dispersion of the coordinates, it is expedient to use the following theorem which can be proved by mathematical induction.

Theorem. When the characteristic polynomial can be written in the form

$$P_n(p) = p^n + a_{n-1}p^{n-1} + \ldots + a_0 = (p + \beta)\, Q_{n-1}(p) \tag{2.1}$$

$$(Q_{n-1}(p) = p^{n-1} + b_{n-2}p^{n-2} + \ldots + b_0)$$

the leading Hurwitz determinants Δ_n^P and Δ_{n-1}^Q are connected by the relation

$$\Delta_n^P = \beta \Delta_{n-1}^Q Q_{n-1}(\beta)$$

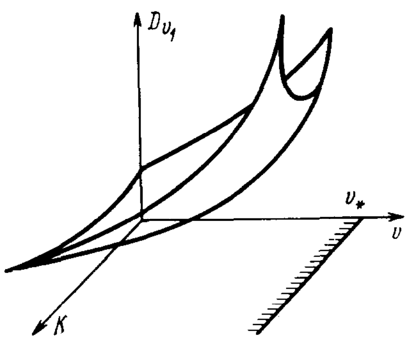

Fig. 2

We note that the polynomial $P_n(p)$ as written in the form (2.1) is encountered fairly often in the statistical dynamics. This stems from the fact that, as a rule, $Q_{n-1}(p)$ represents a characteristic polynomial of the system and the factor $(p + \beta)^{-1}$ appears in the expression for spectral density of the perturbing influence.

In the present case we have

$$Q_{n-1}(p) = p^2 + p\,(A + C) + AC + B - B^2,$$

$$\Delta_{n-1}^Q = (A + C)\,(AC + B - B^2)$$

and from this follows

$$D_{v_1} = \frac{R_\xi(0)\,(1 + k)^2\beta\,(F + HC_1^2)}{2F\,(H - \beta)\,[\beta^2 + \beta\,(A + C) + AC + B - B^2]} \tag{2.2}$$

$$F = \beta\,[AC + B - B^2], \quad H = A + C + \beta$$

When the conditions of stability hold, the dispersion is positive. Considering the expression (2.2) as a function of various parameters we can determine the values of the parameters for which this function attains its minimum value. If we take the coefficient k as such a parameter, then the dispersion D_{v_1} reaches its minimum when $k = k^*$ where

$$k^* = \frac{-(B-1)^2 d_1 d_2 H - HC\,(B-1)\,(d_2 + d_1) - (F + HC^2)}{F + H\,[C + (B-1)\,d_2]^2}$$

Similarly

$$D_{v_2} = \frac{R_\xi(0)\,(d_1 + kd_2)^2\beta\,(F + HC_2^2)}{F\,(H - \beta)\,[\beta^2 + \beta\,(A + C) + AC + B - B^2]}$$

has a minimum when $k = k^{**}$ where

$$k^{**} = \frac{l_1 l_2 (F + A^2 H) + ABH (l_2 - l_1) + B^2 H}{F l_2^2 + H (A l_2 - B)^2}$$

Figure 2 depicts the surface $D_{v_1} = f(k, v)$. When the velocities of motion approach the critical value v^* defined by the expression (1.2), the dispersion increases without bounds as the value of the leading Hurwitz determinant of the system appearing in the denominator of the expression for dispersion tends to zero.

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ON THE CONSTRUCTION OF PLANE STATIONARY SOLUTIONS OF EQUATIONS FOR NONEQUILIBRIUM MAGNETIZED PLASMA

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A method for determining stationary two-dimensional distribution of the electric current and electron temperature in nonequilibrium magnetized plasma is developed with heat conduction and convestion taken into account. Solution is derived in the form of asymptotic expansions in a small parameter. Derivation of the zero approximation for the external and internal expansions is investigated. The problem of current distribution in a channel with infinite electrodes is considered as an example.

1. If heat conduction and convestion are neglected, the problem of stationary distribution of current in nonequilibrium plasma can be reduced to the problem of continuous media electrodynamics with a nonlinear dependence of electrical conductivity and of the Hall parameter (Ω) on the modulus of the vector of electric current density [1]. In the plane case this problem reduces to a quasi-linear equation of second order for the function of current or electrical potential [2-4] (Eq. (3.1) below). When the Hall parameter exceeds a certain value which coincides with the Hall parameter